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Stability in Lyapunov systems☆

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Abstract

The stability of the stationary point of a Lyapunov system [Malkin IG, *Some Problems in the Theory of Non-linear Oscillations*. Moscow: Gostekhizd; 1956.], which describes the perturbed motion of a dynamical system with two degrees of freedom, is investigated. It is assumed that the characteristic equation of the first approximation of the system has two pairs of pure imaginary roots and that the quadratic part of the integral is not sign-definite. Both the non-resonance case, as well as cases of lower order (second-, third- and fourth-order) resonances are considered. The necessary and sufficient conditions for stability are given in cases when the problem is solved by a combination of the first non-linear terms of normal form.

From the Editorial Board. Professor Andrei Leonidovich Kunitsyn, the well-known specialist in the theory of stability and celestial mechanics, will be 70 years old on the 26th July 2006. He was born in Saratov where he lived until moving to Yuzhno-Sakhalinsk in 1952. In 1954, after finishing middle school with a gold medal, he entered the Moscow Aviation Institute (MAI) and, after completing his studies there, worked in the V. N. Chelomei Experimental Design Bureau where he was engaged in calculations of the trajectories of satellites and spaceships. In 1962, he commenced post-graduate studies at the Moscow Aviation Institute where, under the guidance of G. V. Kamenkov, he investigated the stability of non-linear systems with applications in celestial mechanics and space dynamics, playing an active role in the mechanics seminar headed by V. V. Rumyantsev. In 1966, he defended his candidate's dissertation. From 1967 to 1977, he worked in the Department of Theoretical Mechanics of the Moscow Engineering-Physical Institute where he presented a course in theoretical mechanics and special courses in celestial mechanics, the theory of stability and the theory of non-linear oscillations.

He was actively engaged in scientific research, including research with students, five of whom later defended their candidate's dissertation after graduating from the institute, and two of whom (P. S. Krasil'nikov and V. N. Tkhai) subsequently became Doctors of Science. He actively promoted the development of science and education in Kazakhstan, having prepared seven Candidates of Science, several of whom now manage departments.

He has made a considerable contribution to the theory of the stability of non-linear resonance systems. The results of these investigations, with applications in celestial mechanics and space dynamics, described in his doctoral dissertation (1980) and in the monograph "Some Problems of the Stability of Non-linear Resonance Systems" (Alma-Ata: Gylym; 1990) written in co-authorship with his student L. T. Tashimov, are well known both in this country and abroad. In all, he has published around 100 scientific papers, including three monographs and several university textbooks. He

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continues to be actively engaged in scientific research, working at the Department of Theoretical Mechanics of the Moscow Aviation Institute.

The editorial board and the editorial staff of the Journal of Applied Mathematics and Mechanics, his colleagues and students heartily congratulate him on the occasion of his birthday and wish him robust health and new creative successes.

1. Formulation of the problem

The problem of the stability of the equilibrium position or the steady motion of many dynamical systems reduces to the problem of the stability of the trivial solution of the system of equations (differentiation with respect to time is denoted by a dot)

$$\dot{x} = Px + X(x), \quad X(0) = 0, \quad x \in R^n \tag{1.1}$$

(P is a constant matrix and $X(x)$ is an analytic function of x), which possesses an analytic integral

$$H \equiv H_2(x) + H_3(x) + \ldots = h = \text{const} \tag{1.2}$$

where $H_m(x)$ are forms of the m-th order, $m = 2, 3, \ldots$.

If, among the eigenvalues of the matrix P, there is just a single pair of pure imaginary eigenvalues, system (1.1) is said to be a Lyapunov system.[1] Special cases of such systems are conservative and generalized conservative Hamiltonian systems.

An algorithm for constructing the periodic motion in the neighbourhood of the point of rest $x = 0$ was developed by Lyapunov[2] for such systems. These motions form a single-parameter family adjacent to the point of rest. It has been shown[3] that, in the case of two pairs of pure imaginary roots, two families appear of which one can disappear in a resonance situation, as a result of which resonance motions arise. It has also been shown[4] that cycles are born in the neighbourhood of a stable point of rest for a sign-definite integral (1.2) in an "almost" resonance situation at each level $H(x) = h$. The problem of the stability of the point of rest of system (1.1), when the form H_2 of integral (1.2) is not sign-definite (in the opposite case, integral (1.2) guarantees stability), has not been studied until now (Hamiltonian systems, the stability of which is dealt with in a large number of papers (for example, see the review Ref. 5), are an exception). This paper investigates this problem in the case of a dynamical systems with two degrees of freedom ($n = 4$).

We first note that stability of the trivial solution of system (1.1), as in Hamiltonian systems, is only possible in the critical case, that is, when all the eigenvalues of the matrix P are either pure imaginary or zero. Actually, as was shown in Ref. 6, the existence of a quadratic form in the integral (1.2) (which is necessary for the class of systems being considered) always enables us to represent the linear part of system (1.1) in Hamiltonian form. Then, assuming that the matrix P has no zero eigenvalues, system (1.1) can be written in the complex-conjugate variables z_s and $\bar{z}_s$ as

$$\dot{z}_s = i\lambda_s z_s + \sum Z_s^{(m)}, \quad \dot{\bar{z}}_s = -i\lambda_s \bar{z}_s + \sum \bar{Z}_s^{(m)}; \quad s = 1, 2 \tag{1.3}$$

Here $Z_s^{(m)}$ and $\bar{Z}_s^{(m)}$ are complex-conjugate forms of the m-th order ($m = 2, 3, \ldots$) of the complex-conjugate variables $z_1, z_2, \bar{z}_1, \bar{z}_2$, and $\pm i\lambda_s$ are the pure imaginary eigenvalues of the matrix P. It is well known[2,7,8] that the problem of the stability of the trivial solution of system (1.3) depends very much on the arithmetic properties of the quantities λ_1 and λ_2, that is, whether or not they satisfy one of the integral relations

$$\lambda_1 = \lambda_2, \quad \lambda_1 = 2\lambda_2, \quad \lambda_1 = 3\lambda_2 \tag{1.4}$$

which are called the second-, third- and fourth-order internal resonance conditions respectively.[5,7] It is precisely in these cases of resonance that the stability problem for system (1.3) can already be solved using second-order forms or a combination of second- and third-order forms of its right-hand side, while higher-order resonances are only important in the case of a high degree of degeneracy of the right-hand sides of system (1.3) and, when there is no such degeneracy, the problem reduces to the non-resonance case.[7]

2. The non-resonance case and the case of third-order resonance

We will first consider the non-resonance case and the case of third-order resonance as the simplest cases. Normalizing system (1.3) in accordance with what has been indicated earlier,[7] in the polar coordinates $r_s^{1/2}$, $\theta_s(z_s = r_s^{1/2}\exp(i\theta_s)$, $\bar{z}_s = r_s^{1/2}\exp(-i\theta_s))$, we obtain the following normal form for it in the non-resonance case, apart from the third-order terms of the original system inclusive:

$$\dot{r}_s = 2r_s(A_{s1}r_1 + A_{s2}r_2) + \ldots, \quad \dot{\theta}_s = \lambda_s + (B_{s1}r_1 + B_{s2}r_2) + \ldots; \quad s = 1, 2 \tag{2.1}$$

(A_{sj} and B_{sj} are real constants). We shall call the system obtained from (2.1) by discarding the terms which have not been written out the model system (it is obtained from the combination of non-linear terms of system (1.3) up to the third-order inclusive[7])

Taking account of the fact that integral (1.2) in the new variables assumes the form

$$r_1 - r_2 + \ldots = h \tag{2.2}$$

we conclude that $A_{11} = A_{22} = 0$, $A_{12} = A_{21} = A$ in system (2.1). Moreover, in the equations for r_s in their normal form, there will be no terms of the form $A_s^{(m)}r_s^m$ for all $m > 2$.

We will now show that, when $A > 0$, the trivial solution of the original problem (1.3) is unstable. It is easy to show this by considering the behaviour of system (2.1) in the manifold $h = 0$, where $r_1 = r_2 + \ldots$. In fact, putting $r_1 = r_2 = r$, for the model system corresponding to (2.1), we shall have

$$\dot{r} = 2Ar^2$$

and the instability of the system obtained when $A > 0$ is obvious. We know[7] that instability of the trivial solution of the original system also follows from such instability (the existence of an increasing solution).

In the case when $A < 0$, the Lyapunov function

$$V = r_1 + r_2$$

guarantees the stability of the trivial solution of the original system. In fact, by virtue of system (2.1), for its derivative we shall have

$$\dot{V} = 4r_1r_2[A + W(r_1, r_2, \theta_1, \theta_2)]$$

where $W(r_1, r_2, \theta_1, \theta_2)$ is a 2-periodic function in θ_1 and θ_2 which vanishes when $r_1 = r_2 = 0$. Hence, $V^\bullet$ is a sign-definite negative function for the complete system (2.1), and this means that the function V itself satisfies all the conditions of Lyapunov's theorem on the stability of the trivial solution of system (2.1) and, consequently, of the original system (1.1). In the degenerate case when $A = 0$, it is only possible to demonstrate the stability of the trivial solution of the model system corresponding to system (2.1) since, in this case, the sign definiteness of the derivative $V^\bullet$ can be violated when higher-order terms are taken into account. The treatment of this case is beyond the scope of the above-formulated problem of obtaining the stability conditions using the first non-linear terms. We further note that, in the manifold $h = 0$, where $r_1 = r_2 + \ldots$ when $A < 0$, the derivative $V^\bullet$ becomes sign-definite and of opposite sign to V. Consequently, on the basis of Lyapunov's theorem, the trivial solution both of system (2.1) as well as of the original system (1.1) turns out to be asymptotically stable in this case – a property which is fundamentally impossible in Hamiltonian systems. Hence, in the non-resonance case, the following theorem holds.

Theorem 1. *In the non-resonance case, the trivial solution of the original system (1.1) is unstable (stable) if the coefficient in its normal form (2.1) $A > 0$ ($A < 0$) and, if the initial perturbations satisfy the condition $h = 0$, the stability becomes asymptotic. In the degenerate case when $A = 0$, the problem of the stability of the original system is not solved by treating just the first (quadratic and cubic) non-linear terms.*

We will now consider the case of third-order resonance when $\lambda_1 = 2\lambda_2$. In this case, when account is taken of integral (2.2), the normal form of system (1.3) can be represented as (a and b are real constants)

$$\begin{aligned} r_1^{\bullet} &= r_2^{\bullet} = 2RQ, \quad \theta^{\bullet} = R(r_1^{-1} + 2r_2^{-1})dQ/d\theta \\ R &= r_1^{1/2}r_2, \quad Q = Q(\theta) = a\cos\theta + b\sin\theta, \quad \theta = \theta_1 + 2\theta_2 \end{aligned} \tag{2.3}$$

When there is no complete degeneracy of its normal form ($a^2 + b^2 \neq 0$), the existence of an integral in it which is sign-definite and linear with respect to r_1 and r_2 is a necessary and sufficient condition for the stability of the trivial solution of system (2.3).[7] Since integral (2.2) is of alternating sign then, when $a^2 + b^2 \neq 0$, we conclude that the trivial solution of system (2.3) is unstable and that this instability is also preserved in the original system (1.1).[7]

In the case of the above-mentioned degeneracy, on carrying out a further normalization we arrive at system (2.1), and, hence, this case of third-order resonance is completely identical to the non-resonance case. On the basis of the above analysis, it is possible to formulate the following theorem.

Theorem 2. *In the case of third-order resonance* ($\lambda_1 = 2\lambda_2$), *the trivial solution of system* (1.1) *is either unstable (when there is no complete degeneracy of its normal form* (2.3)) *or the problem of its stability is solved as in the non-resonance case using Theorem 1.*

3. The case of fourth-order resonance

We will now investigate the stability of the trivial solution of system (1.1) in the case of a fourth-order resonance when $\lambda_1 = 3\lambda_2$. We note that, with respect to its complexity, it is fundamentally different from the third-order resonance which was considered above and, for systems of a general form, it is impossible to obtain a stability criterion in an algebraic form even for the model system.[8,9] However, the problem can be solved practically completely in the case of the Lyapunov systems being considered.

In this case, putting

$$a\cos\theta + b\sin\theta = C\sin\psi, \quad \theta = \theta_1 + 3\theta_2, \quad C = (a^2 + b^2)^{1/2}$$

the normal form of system (1.3) can be written as:

$$\begin{aligned} &r_1^{\bullet} = r_2^{\bullet} = 2(Ar_1r_2 + CR\sin\psi) + \ldots, \quad \psi^{\bullet} = B_1r_1 + B_2r_2 + \\ &+ CR(r_1^{-1} + 3r_2^{-1})\cos\psi + \ldots; \quad R = (r_1r_2^3)^{1/2} \end{aligned} \tag{3.1}$$

where A, B_1, B_2, C are real constants.

In the manifold, $h = 0$ where $r_1 = r_2 + \ldots$, the normalized part of the system takes the form ($B = B_1 + B_2$)

$$r_1^{\bullet} = 2(A + C\sin\psi)r_1^2 + \ldots, \quad \psi^{\bullet} = (B + 4C\cos\psi)r_1 + \ldots \tag{3.2}$$

and is identical to the normal form of a second-order periodic system in the case of a single-frequency fourth-order resonance which has been fully investigated[7] using the results obtained by Kamenkov for the critical case of two zero roots.[10] The detailed analysis carried out in Ref. 7 showed that, when the inequality

$$4C < |B| \tag{3.3}$$

is satisfied, the stability or instability of the trivial solution of system (3.2) is determined by the sign of the coefficient A: when $A < 0$, there is asymptotic stability and, when $A > 0$, instability. This conclusion also holds in the case of the trivial solution of the original system (1.1) when $h = 0$.

We will now consider the case $A \leq 0$ when $h \neq 0$. Noting that, when $A = 0$, the model system corresponding to (3.1) becomes reversible, we will use the approach developed earlier in Refs. 11,12 and in the case when $A \neq 0$. For this purpose, we consider the function[11]

$$V = (r_1 - r_2)^2 + V_1^2(r_1, r_2, \psi) \tag{3.4}$$

where

$$V_1 = B_1 r_1^2 + B_2 r_2^2 + 4CR\cos\psi$$

Suppose V_1^* is the value of V_1 in the manifold $h=0$. It is then obvious that function (3.4) will be sign-definite in this manifold if

$$V_1^* \equiv (B + 4C\cos\psi) r_1^2 \neq 0, \quad B = B_1 + B_2$$

From this we obtain the condition for function (3.4) to be sign-definite in the form of inequality (3.3).

By virtue of the model system corresponding to (3.2), for the derivative $V^\bullet$ we obtain

$$V^\bullet = 8Ar_1 r_2 V_1^{*2} \tag{3.5}$$

It can be seen that, when $A<0$, the derivative $V^\bullet$ is sign-constant negative and, when $A=0$, it is identically equal to zero. Hence, when inequality (3.3) is satisfied and $A \leq 0$, function (3.4) satisfies all the conditions of Lyapunov's theorem on the stability of the trivial solution of the model system being considered in the manifold $h=0$.

We shall show that, when $A<0$, the derivative (3.5) will also be negative in the vicinity of the above-mentioned manifold and, consequently, the conclusion regarding stability also holds for any perturbations when $h \neq 0$. To do this, we consider the domain

$$G = \{r_1 r_2 \colon r_1 + r_2 \leq 2p, |h| \leq h_0\}$$

(p and h_0 are positive numbers). On the boundary of the domain G, where $r_1 + r_2 = 2p$, $|h| \leq h_0$, we have $2r_2 + h = 2p$. This means that, for small h_0, the derivative of V on this boundary is close to the value (3.5), that is, it remains negative. Hence, the inequality $A \leq 0$ together with (3.3) also remains the (non-asymptotic) stability condition when $h \neq 0$, but only in the case of the model system corresponding to (3.1).

We will now consider the case when, instead of inequality (3.3), we have

$$|B| \leq 4C \tag{3.6}$$

According to the results obtained earlier in Ref. 7, the inequality

$$A < -(C^2 - B^2/4)^{1/2} \tag{3.7}$$

is the necessary and sufficient condition for the asymptotic stability of the trivial solution of system (3.2) in this case. The same conclusion also holds for the initial system when $h=0$. Note that the inequality which is the opposite of (3.7) will also be the condition for the instability of the original system when $h \neq 0$.

We will now show that condition (3.7) also remains the (non-asymptotic) condition for the stability of the trivial solution of the model system corresponding to (3.1) when $h \neq 0$. To do this, on eliminating the variable r_2 in system (3.1) using integral (2.2), we consider the system

$$\begin{aligned} h^\bullet &= 0 \\ r_1^\bullet &= 2[Ar_1(r_1+h) + Cr_1^{1/2}(r_1+h)^{3/2}\sin\psi)] + \ldots \\ \psi^\bullet &= B_1 r_1 + B_2(r_1+h) + CR[r_1^{-1} + 3(r_1+h)^{-1}]\cos\psi + \ldots \end{aligned} \tag{3.8}$$

When $h=0$, the resulting system becomes system (3.2), the trivial solution of which is asymptotically stable when conditions (3.6) and (3.7) are simultaneously satisfied. This follows from the existence of a Lyapunov function $V(r_1, \psi) > 0$[10] with a derivative $V^\bullet(r_1, \psi) < 0$ which is sign-definite. On considering the function $V_1 = h^2 + V$, for its derivative $V_1^\bullet(r_1, \psi, h)$ by virtue of system (3.8) we obtain the function which is continuous with respect to the parameter h and is transformed into $V(r_1, \psi) < 0$ when $h=0$. Then function $V_1^\bullet(r_1, \psi, h)$ will also be negative for sufficiently small h and, consequently, all the conditions of Lyapunov's theorem are satisfied in the case of system (3.8) which means that the trivial solution $r_1 = r_2 = 0$ of system (3.1) will also be stable.

The following theorem is therefore found to hold in the case of a fourth-order resonance.

Theorem 3. *When inequality (3.3) is satisfied, it is necessary and sufficient for the stability of the trivial solution of the model system corresponding to (3.1) that $A \le 0$ (when $A < 0$ and $h = 0$, the stability becomes asymptotic). When the opposite inequality to (3.3) is satisfied, it is necessary and sufficient for the stability of the trivial solution of system (3.1) that the inequality*

$$A < -(C^2 - B^2)^{1/2}/4$$

should be satisfied (the stability becomes asymptotic when $h = 0$). The conditions for instability and asymptotic stability also hold in the case of the original system (1.1).

4. The case of second-order resonance

We will now consider the last case of resonance when $\lambda_1 = \lambda_2$. Here, as in the case of fourth-order resonance, an algebraic criterion does not exist for a system of general form and it is only possible to obtain sufficient conditions for stability or instability.[13] However, the problem is practically completely solved in the case of the Lyapunov systems which are being considered.

In this case, the normal form of system (1.3) will be

$$\begin{aligned}
\dot{r}_1/2 &= R[r_1(Q_1 + Q_2) + r_2 Q_3 + RQ_7] + r_1(A_{11} r_1 + A_{12} r_2) + \dots \\
\dot{r}_2/2 &= R[r_1 Q_4 + r_2(Q_5 + Q_6) + RQ_8] + r_2(A_{21} r_1 + A_{22} r_2) + \dots \\
\dot{\theta} &= B_1 r_1 + B_2 r_2 + R(Q_5' - Q_1' - Q_2' - Q_6') + \\
&+ r_1(r_1/r_2)^{1/2} Q_4' + r_2(r_2/r_1)^{1/2} Q_3' + r_1 Q_8' + r_2 Q_7' + \dots
\end{aligned} \tag{4.1}$$

Here,

$$R = (r_1, r_2)^{1/2}, \quad \theta = \theta_2 - \theta_1$$

$$Q_i = Q_i(\theta) = a_i \cos\theta + b_i \sin\theta, \quad i = 1, \dots, 6$$

$$Q_j = Q_j(\theta) = a_j \cos 2\theta + b_j \cos 2\theta, \quad j = 7, 8$$

a_s and b_s are constants and differentiation with respect to θ is denoted by a prime.

It follows from the existence of integral (2.2) that

$$Q_1 + Q_2 = Q_4, \quad Q_6 + Q_5 = Q_3, \quad Q_7 = Q_8$$

$$A_{11} = A_{22} = 0, \quad A_{12} = A_{21} = A$$

Although the normal form of the system is considerably more complex in this case than in all the preceding cases, it has not fundamentally changed, and this enables us to solve the stability problem without modifying the approach described above. To do this, we again consider the behaviour of system (4.1) on the manifold $h = 0$, where $r_1 = r_2 + \dots$. Then, instead of (4.1), we obtain the following system

$$\begin{aligned}
\dot{r}_1/2 = \dot{r}_2/2 &= [A + C_1 \cos(\theta + \psi_1) + C_2 \cos(2\theta + \psi_2)] r_1^2 + \dots \equiv P(\theta) r_1^2 + \dots \\
\dot{\theta} &= [B_1 + B_2 + C_3 \cos(\theta + \psi_3) + C_4 \cos(2\theta + \psi_4)] r_1 + \dots \equiv Q(\theta) r_1 + \dots
\end{aligned} \tag{4.2}$$

Here $\psi_1, \dots, \psi_4$, $C_1, \dots, C_4$ are constants which are expressed in terms of a_s and b_s in a known way.

It can be seen that the normalized part of system (4.2) has basically the same structure as system (3.2) and, consequently, the conclusions in Ref. 7, which were obtained from Kamenkov's results[10] again hold for it. However, on account of the complexity of the normal form compared with the case of fourth-order resonance, it is not possible here to obtain the stability conditions in an analytical form. However, it is quite feasible to check them for each specific system. As in the case of a fourth-order resonance, it is necessary to distinguish two cases depending on whether the inequality

$$|B_1 + B_2| > C_3 + C_4 \tag{4.3}$$

is satisfied or violated.

The following theorem therefore holds in this case.

Theorem 4. *Suppose inequality (4.3) is satisfied in the case of system (4.2). Then, if* $A > 0$, *its trivial solution (and, also, the trivial solution of the original system) is unstable and, when* $A < 0$, *it is asymptotically stable. When equality (4.3) is violated, the trivial solution of system (4.2) (as well as the trivial solution of the original system) is unstable if the function* $P(\theta)$ *is positive for all solutions of the equation* $Q(\theta) = 0$ *and asymptotically stable if it is negative (in this case, the trivial solution of the original system will be non-asymptotically stable).*

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References

1. Malkin IG. *Some Problems in the Theory of Non-linear Oscillations*. Moscow: Gostekhizdat; 1956.
2. Lyapunov AM. *The General Problem of the Stability of Motion. Collected Papers*. Moscow and Leningrad: Izd. Akad Nauk SSSR; 1956; Vol 2; pp. 7–256.
3. Tkhai VN. Resonant Lyapunov families of periodic motions of reversible systems. *Prikl Mat Mekh* 2004;**68**(3):384–401.
4. Tkhai VN. Cycle in a near-resonance system. *Prikl Mat Mekh* 2004;**68**(2):254–72.
5. Kunitsyn AL, Markeyev AP, Stability in resonance cases. In *Advances in Science and Technology. General Mechanics*. Moscow: VINITI; 1979; Vol. 4, pp. 58–139.
6. Kozlov VV. Linear systems with a quadratic integral. *Prikl Mat Mekh* 1992;**156**(6):900–6.
7. Kunitsyn AL, Tashimov LT. *Some Problems of the Stability of Non-linear Resonant Systems*. Alma-Ata: Gylym; 1990.
8. Khazin LG, Shnol' EE. The simplest cases of algebraic non-solvability in problems of asymptotic stability. *Dokl Akad Nauk SSSR* 1978;**240**(6):1309–11.
9. Krasil'nikov PS. Asymptotic stability accompanying 1:3 resonance. *Prikl Mat Mekh* 1996;**60**(1):23–9.
10. Kamenkov GV. Two roots equal to zero. In: Kamenkov GV, editor. *Stability of Motion. Oscillations Aerodynamics Selected Papers*, vol. 1. Moscow: Nauka; 1971. p. 39–84.
11. Tkhai VN. Stability of mechanical systems under the action of positional forces. *Prikl Mat Mekh* 1980;**44**(1):40–8.
12. Tkhai VN. Reversibility of mechanical systems. *Prikl Mat Mekh* 1991;**55**(4):578–86.
13. Krasil'nikov PS. A generalized scheme for the construction of Lyapunov functions from first integrals. *Prikl Mat Mekh* 2001;**65**(2):199–210.

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